A CHARACTERIZATION OF THE EUCLIDEAN TOPOLOGY AMONG THE AFFINE TOPOLOGIES

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ABSTRACT

Theorem: The Euclidean topology on a finite dimensional vector space X is the weakest Hausdorff affine topology on X for which X is second category in itself.

Introduction

A topology Γ on a real vector space X is called an affine topology provided (i) for each element y of X and each real number a the map $T: X \to X$ defined by $Tx = ax + y$ is continuous, (ii) for each element y of X (y distinct from the neutral element θ of X) the map $S: R \to X$ defined by $Sa = ay$ is a homeomorphism (where R denotes the set of real numbers with the usual topology) and (iii) for each element y of X the set $\{ay: a \in R\}$ is closed. This concept was introduced by M. Frechet [2, p. 203] and has been studied by V. Klee [3, 4, 5]. Every linear topological space (in the sense of Dunford and Schwartz [1]) in which points are closed has an affine topology, but not every affine topology makes a space a linear topological space. The concepts coincide only on one-dimensional spaces.

The purpose of this paper is to point out a topological feature of the Euclidean topology on a finite dimensional vector space which distinguishes it among the other affine topologies. A statement of this characterization is found in the abstract and is to be interpreted in the strongest sense: for each Hausdorff affine topology Γ which is stronger than the Euclidean topology on X it is true that (X, Γ) is second category, and for each Hausdorff affine topology Γ which is either strictly weaker than or not comparable to the Euclidean topology on X it is true that

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(X, Γ) is first category. The proof is contained in the statements of Lemmas 4 and 5.

An example at the end of the paper shows that the theorem is not vacuous by exhibiting a Hausdorff affine topology strictly weaker than the Euclidean topology on each n-dimensional space ($1 < n < \infty$). Many examples of affine topologies which are strictly stronger than the Euclidean topology (and hence necessarily Hausdorff) are known, for example, the topology Γ_0 discussed below (also see [5]).

In the following, whenever a topolgical term (for example: closed, interior) is used without being modified by a specific topology (for example: Γ-closed, Γ_0-interior) it refers to the Euclidean topology.

The proof

Let A be a subset of a real vector space X and let x be an element of A. We say A is radial at x provided for each y in X there is an $\varepsilon > 0$ such that $\{x + ty: 0 \leqq t \leqq \varepsilon\}$ is a subset of A. It is not difficult to see that if a set A is open in an affine topology then A is radial at each point of A. Thus the strongest affine topology a vector space may possess is that in which precisely those sets which are radial at each of their points are open. This topology is usually called the core topology and we denote it by Γ_0.

Lemmas 1 and 2 are interesting because they reveal relations between the Γ_0-interior and the interior of subsets of finite dimensional vector spaces. Other relations of this type were studied in [3]. The two-dimensional case of Lemma 1 follows from a result of Klee [5, Proposition 2].

LEMMA 1. *Let A be a Γ_0-open subset of an n-dimensional vector space X ($n < \infty$). Then Γ_0-closure (A) has interior.*

PROOF. The statement is trivial if $n = 1$. Assume the lemma to be true for $(n-1)$-dimensional vector spaces. We can further assume $\theta \in A$. Let $\{x_1, \cdots, x_n\}$ be a basis for X. Choose a number $a > 0$ so that $B = \{tx_1 : 0 \leqq t \leqq a\}$ is a subset of A. For each $y \in B$ let

$$S_y = \{y + t_2x_2 + \cdots + t_nx_n : -1 < t_i < 1,\ i = 2, \cdots, n\}.$$

Since $(A \cap S_y) - y$ is a Γ_0-open set in the linear span of $x_2, \cdots, x_n$, the induction hypothesis assures the existence of an integer $k(y)$ and real numbers $s_2(y), \cdots, s_n(y)$ such that the $(n-1)$-cube $C_y = \{y + t_2x_2 + \cdots + t_nx_n : s_i(y) \leqq t_i \leqq s_i(y) + 1/k(y),\ i = 2, \cdots, n\}$ is contained in Γ_0-closure $(A \cap S_y)$. Since B is second category, there is an integer k such that $D = \{y \in B: k(y) = k\}$ is dense on some subinterval of B.

For each $i = 2, \cdots, n$ let $P_i = \{-1, -1 + 1/2k, -1 + 2/2k, \cdots, +1 - 1/2k\}$ and let Q be the Cartesian product $\prod_{i=2}^{n} P_i$ so that the cardinality of Q is $(4k)^{n-1}$. Define $f: D \rightarrow Q$ by $f(y) = (r_2, \cdots, r_n)$ provided $r_i \leqq s_i(y) < r_i + 1/2k$. The preimages of points in Q partition D into at most $(4k)^{n-1}$ subsets. Hence some element in this partition, say $f^{-1}(r_2, \cdots, r_n)$ is dense in some open subinterval E of B. (It is an easy exercise to show that if W is dense in some interval of the real line and $\mathscr{P}$ is a finite partition of W then some element of $\mathscr{P}$ is dense in some interval.) We conclude that for each y in a dense subset of E $\{y + t_2x_2 + \cdots + t_nx_n: r_i + 1/2k < t_i < r_i + 2/2k, i = 2, \cdots, n\}$ is a subset of Γ_0-closure (A). Thus

$$\{t_1x_1 + \cdots + t_nx_n: t_1x_1 \in E, r_i + \frac{1}{2k} < t_i < r_i + \frac{2}{2k}, i = 2, \cdots, n\}$$

is an open set contained in Γ_0-closure (A).

LEMMA 2. *Let Γ be a Hausdorff affine topology on a finite dimensional vector space X and let x and y be distinct elements of X. Then there exist a Γ-open set V containing x and an open set U containing y such that $U \cap V = \varnothing$.*

PROOF. We may assume that $x = \theta$. Let V_1 and V_2. be disjoint Γ-open Γ neighborhoods of θ and y respectively. Replacing V_1 by $V_1 \cap (V_2 - y)$ and V_2 by $V_2 \cap (V_1 + y)$ if necessary, we may assume that $V_2 = V_1 + y$. The topology Γ is weaker than Γ_0, therefore by Lemma 1, V_1 is Γ-dense in some open set W. Let $z \in W \cap V_1$ and define $V = V_1 - z$ and $U =$ interior (Γ-closure $(V_2 - z)$). Now V is a Γ-open set containing θ and U is an open neighborhood of y. Furthermore, U and V are disjoint since $(V_1 - z) \cap (\Gamma\text{-closure } (V_2 - z)) = \varnothing$.

LEMMA 3. *If Γ is a Hausdorff affine topology on a finite dimensional vector space X and A is a compact subset of X then A is Γ-closed.*

PROOF. Let $x \in X$ with $x \notin A$. By Lemma 2, for each $a \in A$ there exist disjoint sets V_a and U_a such that V_a is a Γ-neighborhood of x and U_a is a neighborhood of a. Choose $\{a_1, \cdots, a_n\} \subset A$ such that $A \subset \cup \{U_{a_i}: i = 1, \cdots, n\}$. It follows that $\cap \{V_{a_i}: i = 1, \cdots, n\}$ is a Γ-neighborhood of x disjoint from A and thus A is Γ-closed.

LEMMA 4. *If Γ is a Hausdorff affine topology on a finite dimensional vector space X which is either strictly weaker than or not comparable to the Euclidean topology on X, then (X, Γ) is first category in itself.*

PROOF. Since Γ is not stronger than the Euclidean topology there can exist no non-empty Γ-open sets which are bounded. (If there were a bounded non-empty Γ-open set there would be a bounded Γ-neighborhood of θ and, since neighborhoods of θ absorb each bounded set, it would follow that Γ is stronger than the Euclidean topology.) Thus compact sets have no Γ-interior and it follows from Lemma 3 that compact sets are Γ-nowhere dense. But every finite dimensional space may be covered by a countable number of compact sets.

We remark that Lemmas 3 and 4 are false if Γ is allowed to be a non-Hausdorff affine topology.

LEMMA 5. *If Γ is an affine topology on an n-dimensional vector space X $(n < \infty)$ which is stronger than the Euclidean topology on X, then (X,Γ) is second category in itself.*

PROOF. Let $\{F_1, F_2, \cdots\}$ be a sequence of Γ-closed subsets of X such that $X = \cup\{F_i: i = 1,2,\cdots\}$. We will show that for each set U in X with non-empty interior there is an F_j which contains an open (and therefore Γ-open) subset of U and hence is not Γ-nowhere dense. This statement is clear when $n = 1$, and by induction we assume it is true for spaces of dimension $n - 1$. Let $\{x_1, \cdots, x_n\}$ be a basis for X. It suffices to consider $U = \{t_1x_1 + \cdots + t_nx_n: -1 < t_i < 1$ for $i = 1, \cdots, n\}$. Let $B = \{tx_1: -1 < t < 1\}$ and for each $y \in B$ let $S_y = \{y + t_2x_2 + \cdots + t_nx_n: -1 < t_i < 1$ for $i = 2 \cdots, n\}$. By the proper translation, S_y may be considered an open subset of an $(n-1)$-dimensional space and hence, by induction, there exist integers $j(y)$ and $k(y)$ and real numbers $s_2(y), \cdots, s_n(y)$ such that $C_y = \{y + t_2x_2 + \cdots + t_nx_n: s_i(y) < t_i < s_i(y) + 1/k(y)$ for $i = 2, \cdots, n\}$ is a subset of $F_{j(y)} \cap S_y$. Since B is second category, there exist integers j and k such that $D = \{y \in B: j(y) = j$ and $k(y) = k\}$ is dense on some subinterval of B. Now an argument identical to that found in the second paragraph of the proof of Lemma 1 (with Γ_0-closure (A) replaced by F_j) shows F_j contains an open subset of U.

An example

We will construct an example of a Hausdorff affine topology Γ_1 on each n-dimensional vector space X $(n \geqq 2)$ which is strictly weaker than the Euclidean topology. This will be done with the aid of the following proposition which perhaps is of interest in itself.

PROPOSITION. *There exists a collection $\mathfrak{I}$ of open subsets of the set of real numbers such that*

(a) *if* $F \in \mathfrak{F}$ *and* r *is a non-zero real number then* $rF \in \mathfrak{F}$,

(b) *if* $F, G \in \mathfrak{F}$ *then* $F \cap G \in \mathfrak{F}$,

(c) *if* $F \in \mathfrak{F}$ *and* $x \in F$ *then* $F - x \in \mathfrak{F}$,

(d) *if* x *and* y *are distinct real numbers there is an* $F \in \mathfrak{F}$ *such that* $x + F$ *and* $y + F$ *are disjoint, and*

(e) *each* $F \in \mathfrak{F}$ *is unbounded and contains* 0.

PROOF. Let $F_0 = \cup\{(2n - \frac{1}{2}, 2n + \frac{1}{2})$: n is an integer$\}$ and define $\mathfrak{F} = \{\bigcap_{i=1}^{m} s_i(F_0 - t_i)$: s_i and t_i are real numbers with $s_i \neq 0$, $t_i \in F_0$ for $i = 1, \cdots, m$ and m is a positive integer$\}$. Let $F \in \mathfrak{F}$, r be a non-zero real number and $x \in F$. For some s_i and t_i we have $F = \cap\{s_i(F_0 - t_i): i = 1, \cdots, n\}$, therefore $rF = \cap\{rs_i(F_0 - t_i): 1, \cdots, n\}$. Thus $\mathfrak{F}$ satisfies (a). $\mathfrak{F}$ clearly satisfies (b). To prove $\mathfrak{F}$ satisfies (c) we must show $F - x \in \mathfrak{F}$. But $F - x = \cap\{s_i(F_0 - t_i) - x: i = 1, \cdots, n\}$ and hence, by (b), it suffices to show $s_i(F_0 - t_i) - x \in \mathfrak{F}$ for $i = 1, \cdots, n$. Since $x \in s_i(F_0 - t_i)$ it follows that $t_i + x/s_i \in F_0$, hence $s_i(F_0 - t_i) - x = s_i(F_0 - (t_i + x/s_i)) \in \mathfrak{F}$. If x and y are distinct real numbers then $(x + (x - y)F_0) \cap (y + (x - y)F_0 = \varnothing$, therefore $\mathfrak{F}$ satisfies (d). It is clear that each $F \in \mathfrak{F}$ contains 0 and it remains only to show $F = \cap\{s_i(F_0 - t_i): i = 1, \cdots, n\}$ is unbounded. We will establish this by showing F contains infinitely many integers. Let Z and 2Z denote the integers and the even integers respectively. For each real number r define $[r]$ to be the equivalence class of r in the quotient group $R/2Z$. Define $f: Z \to (R/2Z)^n$ by $f(k) = ([t_1 + k/s_1], \cdots, [t_n + k/s_n])$. By a standard compactness argument there exist infinitely many integers k such that $[t_i + k/s_i] \in \cup\{[\varepsilon]: -\frac{1}{2} < \varepsilon < \frac{1}{2}\}$ for each $i = 1, \cdots, n$. But for each i, $[t_i + k/s_i] \in \cup\{[\varepsilon]: -\frac{1}{2} < \varepsilon < \frac{1}{2}\}$ implies $t_i + k/s_i \in F_0$ which in turn implies $k \in s_i(F_0 - t_i)$. Therefore there are infinitely many integers k such that $k \in F$.

EXAMPLE. *A Hausdorff affine topology* Γ_1 *on* X *which is strictly weaker than the Euclidean topology on* X. Let $\{x_1, \cdots, x_n\}$ be a basis for X and let $\mathfrak{F}$ be a collection as in Proposition 1. For each $\varepsilon > 0$ and $F \in \mathfrak{F}$ define $G(\varepsilon, F) = \{t_1x_1 + \cdots + t_nx_n : t_1 \in F$ and either $|t_i| < \varepsilon$ for $i = 1, \cdots, n$ or $|t_1| \geqq \varepsilon$ and $0 < |t_i| < \varepsilon$ for $i = 2, \cdots, n\}$, and for each $x \in X$ define $G(x, \varepsilon, F) = G(\varepsilon, F) + x$. Define Γ_1 as the topology whose base is $\{G(x, \varepsilon, F): x \in X, \varepsilon > 0$, and $F \in \mathfrak{F}\}$. Properties (a), (b), and (c) of Proposition 1 are used to show Γ_1 is an affine topology on X, and properties (d) and (e) show respectively that Γ_1 is Hausdorff and strictly weaker than the Euclidean topology.

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